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Research Notes

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Inertial-Range Spectrum of Hydromagnetic Turbulence

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This note suggests that Kolmogorov's concept of independence of widely separated wavenumbers in the inertial range of turbulence should be modified for the hydromagnetic case. When the magnetic energy in sub-inertial wavenumbers exceeds the total energy in the inertial range, the predicted asymptotic inertial-range energy spectrum is proportional to $k^{-3/2}$, instead of $k^{-5/3}$, and displays exact equipartition between magnetic and kinetic energy. Generalized Lagrangian functions for hydromagnetic flow are introduced to provide a suitable basis for quantitative turbulence approximations. The proposed formalism permits following the fate of given Alfvén wavefronts as well as that of given fluid particles.

Let the hydromagnetic equations in Elsasser's symmetrized form be

$$\left(\frac{\partial}{\partial t} + \mathbf{z}\cdot\nabla\right)\mathbf{w} = \nu_+\nabla^2\mathbf{w} + \nu_-\nabla^2\mathbf{z} - \nabla p, \qquad \nabla\cdot\mathbf{w} = 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{w}\cdot\nabla\right)\mathbf{z} = \nu_+\nabla^2\mathbf{z} + \nu_-\nabla^2\mathbf{w} - \nabla p, \qquad \nabla\cdot\mathbf{z} = 0, \quad (2)$$

where

$$\mathbf{w} = \mathbf{u} + \mathbf{b}, \qquad \mathbf{z} = \mathbf{u} - \mathbf{b},$$
$$\nu_+ = \tfrac{1}{2}[\nu + (4\pi\mu\sigma)^{-1}], \quad \nu_- = \tfrac{1}{2}[\nu - (4\pi\mu\sigma)^{-1}], \quad (3)$$

$\mathbf{u}(\mathbf{x}, t)$ is the Eulerian velocity field, $(4\pi\mu\rho)^{1/2}\,\mathbf{b}(\mathbf{x}, t)$ is the induction field, $p(\mathbf{x}, t)$ is total kinematic pressure, ν is kinematic viscosity, μ is susceptibility, ρ is fluid density, and σ is conductivity.[1]

These equations show an essential difference between the dynamical effects of uniform velocity and magnetic fields. Suppose that pre-existing fields $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{b}(\mathbf{x}, t)$ are augmented by switching on a constant and uniform velocity field $\bar{\mathbf{u}}$. The effect is to replace $\partial/\partial t$ in both (1) and (2) by the substantial derivative $\partial/\partial t + \bar{\mathbf{u}}\cdot\nabla$. This expresses the fact that in a coordinate system moving with $\bar{\mathbf{u}}$ the dynamics of the pre-existing fields are unchanged. If, instead, a uniform magnetic field $\bar{\mathbf{b}}$ is switched on, $\partial/\partial t$ in (1) is replaced by $\partial/\partial t - \bar{\mathbf{b}}\cdot\nabla$ while $\partial/\partial t$ in (2) is replaced by $\partial/\partial t + \bar{\mathbf{b}}\cdot\nabla$. There is no coordinate system in which the dynamics are unchanged. The pre-existing $\mathbf{z}$ and $\mathbf{w}$ fields are propagated as Alfvén waves in opposite directions along the lines of force of the $\bar{\mathbf{b}}$ field. The nonlinear terms in (1) and (2) represent a coupling between the $\mathbf{z}$ and $\mathbf{w}$ waves. If the pre-existing fields $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{b}(\mathbf{x}, t)$ are localized in a region of space of dimension L, then in a time of order $L/|\bar{\mathbf{b}}|$, the imposed uniform field separates the initial excitation into noninteracting $\mathbf{w}$ and $\mathbf{z}$ waves which propagate linearly.

Now consider homogeneous, isotropic hydromagnetic turbulence. Let $E(k)$ and $F(k)$ be the kinetic and magnetic energy-spectrum functions, so that the root-mean-square values v_0 and b_0 of velocity and magnetic field along any axis are given by

$$\tfrac{3}{2}v_0^2 = \int_0^\infty E(k)\,dk, \qquad \tfrac{3}{2}b_0^2 = \int_0^\infty F(k)\,dk. \quad (4)$$

Assume that there exists an inertial range of wavenumbers k such that: Almost all of the contribution to v_0^2 and b_0^2 comes from wavenumbers below the range; almost all of the dissipation into heat occurs above the range; the total kinetic and magnetic energies above any wavenumber k in the range are each small compared to both v_0^2 and b_0^2; energy cascade within the range is local in the sense that there is negligible direct transfer of energy between wavenumbers whose ratio is very large. The conditions under which such a range can be produced will not be examined here.

Kolmogorov's inertial-range law for hydrodynamic turbulence is based on an assumption that the action of energy-range excitation on inertial-range excitation is asymptotically a distortion-free convection which does not affect energy transfer within the inertial range.[2] That is, the energy-range excitation acts, in this respect, as if it were a spatially uniform velocity field. In the present hydromagnetic case, it still may be argued plausibly that the action of the energy range on the inertial range is equivalent to that of spatially uniform fields. But, in contrast to a uniform velocity field, a uniform magnetic field has a profound effect on energy transfer. The propagation of the fluctuations in the **z** and **w** fields in opposite directions disrupts phase relations and thereby may be expected, on the average, to decrease energy transfer. The picture of the inertial-range dynamics implied here is of Alfvén waves with inertial-range wavenumbers propagating through the fluid at speeds of order b_0, with the energy cascade resulting from scattering between the **w** and **z** waves.

A first conclusion is that there should be asymptotically exact equipartition of energy in the inertial range, $E(k) = F(k)$. The argument is as follows: Since the two waves propagate in opposite directions, any initial statistical correlation between the **w** and **z** amplitudes for a given inertial-range wave vector **k** is destroyed in a time of order $(b_0k)^{-1}$. Any persistent correlation must arise from the scattering of the **w** and **z** waves. The distortions which the scattering can produce in the time $(b_0k)^{-1}$ are measured by the ratios v_k/b_0 and b_k/b_0, where, say, $v_k = [kE(k)]^{1/2}$, $b_k = [kF(k)]^{1/2}$. By previous assumption, these ratios are very small for inertial-range k. Hence the **w** and **z** waves scatter only weakly. They are nearly freely propagating and nearly uncorrelated. Asymptotically, for an inertial range of infinite extent, the correlation is zero. The argument is completed by noting from (3) that uncorrelated **w** and **z** amplitudes for a given wave vector imply equipartition for that wave vector.

In order to infer the form of the inertial-range spectrum, it is necessary to estimate the magnitude of the triple correlations. Consider a triad of wavenumbers each of which are of order k. If the energy-range excitation were absent, it would be expected that the nonlinear interaction would build up substantial triple correlations in the local dynamical time $(v_kk)^{-1}$. With the energy-range excitation present, however, $(b_0k)^{-1}$ is the effective time for relaxation of the locally built-up phase correlations through propagation. Since $(b_0k)^{-1} \ll (v_kk)^{-1}$, it is plausible that the resultant steady-state triple correlation is $\propto (b_0k)^{-1}$ and that, therefore,

$$\epsilon \propto (b_0k)^{-1}, \tag{5}$$

where ϵ is the total rate of energy-transfer per unit mass from wavenumbers below k to wavenumbers above k. Because energy is conserved by the nonlinear interaction and a local cascade has been assumed, ϵ is independent of k. Moreover, the assumption of local cascade suggests that, apart from the dependence on b_0, ϵ depends only on the local quantities k and $E(k) = F(k)$. Then dimensional analysis yields

$$\epsilon = A^2b_0^{-1}[E(k)]^2k^3, \tag{6}$$

or

$$E(k) = F(k) = A(\epsilon b_0)^{\frac{1}{2}}k^{-\frac{3}{2}}, \tag{7}$$

where A is a numerical constant.

Equation (7) implies that the skewness factors of the spatial derivatives of the fields approach zero as the Reynolds numbers go to infinity. This is another expression of the weakness of phase correlations in the inertial range.

In a discussion of the dissipation range of hydromagnetic turbulence, Moffatt[3] has suggested that sufficiently strong turbulent magnetic excitation at given wavenumbers can act to suppress turbulence at higher wavenumbers. The inertial-range dynamics suggested above represent an opposite situation. There is no dissipation in the ranges considered. The interaction between energy range and inertial range is purely elastic and acts to spoil energy transfer within the latter range. Consequently, there is a pile-up of energy in the inertial range, and the spectrum level there is higher than a pure Kolmogorov cascade would give.

The direct-interaction closure approximation for hydromagnetic turbulence[4-6] starts from the equations of motion and yields a quantitative description of the energy-transfer process which agrees with the qualitative picture above, except for one important difference: The effective relaxation time for triple correlations is given by $(v_0'k)^{-1}$, where

$$v_0' = (v_0^2 + b_0^2)^{\frac{1}{2}}, \tag{8}$$

and consequently the spectrum is given by

$$E(k) = F(k) = A'(\epsilon v_0')^{\frac{1}{2}}k^{-\frac{3}{2}}, \tag{9}$$

where the constant A' is determined by the approximation.

In the pure hydrodynamic case, the relaxation time $(v_0k)^{-1}$ for inertial-range phase correlations is spurious and due to an inability of the direct-interaction approximation properly to describe the convection of small spatial scales by large spatial scales.[2] This deficiency has been corrected by an alteration of the approximation which, in effect, permits following the relaxation of triple correlations in Lagrangian instead of Eulerian coordinates.[7] The resulting equations for hydrodynamic turbulence give a $-5/3$ inertial range and yield a value of the numerical coefficient that is consistent with experiment.

The altered approximation can be extended to the hydromagnetic case so as to remove the similarly spurious appearance of v_0' instead of b_0 in (9). The Lagrangian alteration is called for if $v_0 > b_0$. Moreover, for $v_0 \gg b_0$, there may be situations in which a two-piece inertial range occurs: First, a hydrodynamic range where $E(k) \propto k^{-5/3}$ and $F(k)/E(k)$ increases from very small values as k increases; second, an equipartition range obeying the $-3/2$ law. In this case, the Lagrangian alteration would yield the correct behavior while the unaltered direct-interaction approximation would yield $-3/2$ dependencies in both ranges.

A straightforward extension of Ref. 7 to the hydromagnetic case would be based on the functions $\mathbf{u}(\mathbf{x}, t \mid r)$ and $\mathbf{b}(\mathbf{x}, t \mid r)$ defined by

$$\mathbf{u}(\mathbf{x}, t \mid t) = \mathbf{u}(\mathbf{x}, t), \qquad \mathbf{b}(\mathbf{x}, t \mid t) = \mathbf{b}(\mathbf{x}, t), \tag{10}$$

$$\begin{aligned}[][\partial/\partial t + \mathbf{u}(\mathbf{x}, t)\cdot\nabla]\mathbf{u}(\mathbf{x}, t \mid r) &= 0,\\ [\partial/\partial t + \mathbf{u}(\mathbf{x}, t)\cdot\nabla]\mathbf{b}(\mathbf{x}, t \mid r) &= 0.\end{aligned} \tag{11}$$

These functions give the velocity and magnetic fields measured at time r at a point which moves with the fluid and which passes through $\mathbf{x}$ at time t. They encompass both the Eulerian and Lagrangian functions as usually defined (cf. Ref. 7).

A more appropriate basis for the hydromagnetic case may turn out to be the functions $\mathbf{w}^+(\mathbf{x}, t|r)$, $\mathbf{z}^+(\mathbf{x}, t \mid r)$, $\mathbf{w}^-(\mathbf{x}, t \mid r)$, and $\mathbf{z}^-(\mathbf{x}, t \mid r)$ defined by

$$\begin{aligned}\mathbf{w}^+(\mathbf{x}, t \mid t) = \mathbf{w}^-(\mathbf{x}, t \mid t) &= \mathbf{w}(\mathbf{x}, t),\\ \mathbf{z}^+(\mathbf{x}, t \mid t) = \mathbf{z}^-(\mathbf{x}, t \mid t) &= \mathbf{z}(\mathbf{x}, t),\end{aligned} \tag{12}$$

$$\begin{aligned}[][\partial/\partial t + \mathbf{w}(\mathbf{x}, t)\cdot\nabla]\mathbf{w}^+(\mathbf{x}, t \mid r) &= 0,\\ [\partial/\partial t + \mathbf{w}(\mathbf{x}, t)\cdot\nabla]\mathbf{z}^+(\mathbf{x}, t \mid r) &= 0,\end{aligned} \tag{13}$$

$$\begin{aligned}[][\partial/\partial t + \mathbf{z}(\mathbf{x}, t)\cdot\nabla]\mathbf{w}^-(\mathbf{x}, t \mid r) &= 0,\\ [\partial/\partial t + \mathbf{z}(\mathbf{x}, t)\cdot\nabla]\mathbf{z}^-(\mathbf{x}, t \mid r) &= 0.\end{aligned} \tag{14}$$

These functions give the values of the fields measured at time r at points which pass through $\mathbf{x}$ at time t but which move, relative to the fluid, along the lines of force and with the local Alfvén speed. The plus and minus signs correspond to relative motion parallel and antiparallel to the local magnetic field direction. The functions $\mathbf{u}(\mathbf{x}, t|r)$ and $\mathbf{b}(\mathbf{x}, t|r)$ provide a means of following the history of a given fluid element at it is carried about by the flow. The functions $\mathbf{w}^+(\mathbf{x}, t|r)$, $\mathbf{z}^+(\mathbf{x}, t|r)$, $\mathbf{w}^-(\mathbf{x}, t|r)$, and $\mathbf{z}^-(\mathbf{x}, t|r)$ permit tracing what happens to a given Alfvén wavefront as it simultaneously propagates along the field lines and is convected by the flow. They may prove useful in stability analysis and in other nonturbulent flow problems, as well as in the turbulence problem.

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Frequency Spectrum of Wind-Generated Waves

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An interesting phenomenon in fluid dynamics is the generation of water waves by wind. The properties of wind-generated waves, and the mechanism for their growth have been investigated by workers in many fields from civil engineering to oceanography. It has been found that a wavy surface may be described statistically in part by the spectral energy function $\Phi(f)$, where f denotes the frequency. $\Phi(f)$ represents the Fourier transform of the calculated autocorrelation function for water waves. The average potential energy E contained in the train of waves is proportional to the variance σ^2 of the surface:

$$E \propto \sigma^2 = \int_0^\infty \Phi(f)\, df. \tag{1}$$

Thus, $\Phi(f)$ denotes that part of the total potential energy which corresponds to waves of frequency f.

For a particular frequency band df, with center frequency f, Φ increases with wind speed V, and fetch F up to a certain value. As the wind speed and fetch continue to increase, the high-frequency region of Φ tends to approach an equilibrium distribution. Phillips,[1] on dimensional grounds, found that the equilibrium range for gravity waves has the form

$$\Phi \propto g^2 f^{-5}, \tag{2}$$

where g is the gravitational acceleration. On the other hand, Hicks[2] has suggested that the spectrum for "pure" capillary waves, which should depend only on the ratio of the surface tension T to the mass density ρ and the frequency, can be described as

$$\Phi \propto (T/\rho)^{2/3} f^{-7/3}. \tag{3}$$

Spectral measurements of waves on lakes and on the ocean have produced considerable evidence for the existence of the equilibrium range for gravity waves. However, to the authors' knowledge, no data have given direct verification for the $f^{-7/3}$ range in capillary waves. The spectra of Cox, as calculated by Hicks,[3] suggest that high frequency components differ from the behavior predicted by Eq. (2), but these measurements appear to be inconclusive.

Spectra showing equilibrium behavior are frequently found for ocean waves at moderate conditions of wind speed and fetch. The equilibrium